THE MINIMAL HIRSCH–BROWN MODEL VIA CLASSICAL HODGE THEORY

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Abstract. In our book on cohomological methods in transformation groups the minimal Hirsch–Brown model was used to good effect. The construction of the model there, however, was rather abstract. Here, for smooth compact connected Lie group actions on smooth closed manifolds, we give a much more explicit construction of the minimal Hirsch–Brown model using operators from classical Hodge theory and the small Cartan model.

1. Introduction.

In [A, P], the minimal Hirsch–Brown model is described in detail and used to good effect. (See, for example, [A, P] sections 1.3, 1.4, 4.4 and 4.6.) The construction of the minimal Hirsch–Brown model there, however, is rather abstract. Our purpose here is to give a more explicit construction of the model for smooth compact connected Lie group actions on closed smooth manifolds using operators from classical Hodge theory. Two of our main results, Theorem (3.8) and Corollary (3.9), are particularly nice in view of their relation to [A2].

In Section 2 we introduce our notation, and we give a brief description of the small (Cartan) model that we use for non–abelian Lie groups. Section 3 gives the explicit construction of the minimal Hirsch–Brown model. Section 4, as an example, discusses the familiar product structure in the equivariant cohomology of a Hamiltonian circle action on $\mathbb{C}P^n$, the idea being to compute the deformation of the product (as one goes from ordinary to equivariant cohomology) in terms of the moment map in two different ways.

2. Notation.

Let G be a compact connected Lie group acting smoothly on a closed smooth manifold M. Suppose that M has an invariant Riemannian metric r. If one does Hodge theory with respect to r, then all the usual operators, for example, $*$, d^*, H (the projection onto the harmonic forms) and G (Green's operator), are invariant. Since the Lie group G is connected, it acts trivially on the cohomology of M; and so all harmonic forms are invariant. We shall let $\Omega_{inv}(M)$ or $\Omega(M)^G$ denote the cochain complex of invariant forms. Thus the usual Hodge decomposition ($\alpha = H(\alpha)+dd^*G(\alpha)+d^*dG(\alpha)$) restricts to $\Omega_{inv}(M)$ without alteration. (For an introduction to Hodge theory, see [W].)

In this paper we shall always assume that the Lie group, G, the manifold, M, and the Riemannian metric are as in the paragraph above.

When $G = T^k$, the k–torus, we shall use the Cartan model to compute the equivariant cohomology, $H^*_G(M;\mathbb{R})$. Let $R_G = H^*(BG;\mathbb{R}) = \mathbb{R}[t_1, \ldots, t_k]$, where each t_i has degree 2. There is a canonical association between $t_1, \ldots, t_r$ and a basis $c_1, \ldots, c_r$ of $\mathfrak{g}$, the Lie algebra of G. And, to each c_j, there is associated a vector field V_j on M, via the exponential map, $\mathfrak{g} \to G$, and the group action. The Cartan model is then $R_G \otimes \Omega_{inv}(M)$ with differential $d_G = I \otimes d - \partial$, where $\partial = \sum_{j=1}^{k} t_j \otimes i_j$ and $i_j = i_{V_j}$, the inner product, for $1 \le j \le k$. We usually abbreviate $I \otimes d$ as d; and so $d_G = d - \partial$. (See [A, B] or [G, S] for details.)

For non–abelian G, it is convenient here to use the small Cartan model. This was proposed by a number of authors (see, for example, [G, K, M]), and it has been shown to be equivalent to the Cartan model by Alekseev and Meinrenken ([A, M]). Formally, the small model looks just like the above, that is, $R_G \otimes \Omega_{inv}(M)$ with differential $d - \partial$,

where $R_G = H^*(BG;\mathbb{R}) = \mathbb{R}[t_1, \ldots, t_k]$ and $\partial = \sum_{j=1}^{k} t_j \otimes i_j$. So k is the rank of G. Each i_j is now, however, the inner product with a multivector corresponding to an element $c_j \in (\wedge\mathfrak{g})^G$, where $\{c_1, \ldots, c_k\}$ is a basis of $(\wedge\mathfrak{g})^G$. Each c_j has positive odd degree, but the degrees may be more than one. The polynomial generator t_j corresponds to c_j via a canonical transgression. So deg $t_j =$ deg $c_j + 1$; and, as an operator, deg $i_j = 1 -$ deg t_j. For example, when $G = SU(2)$, $k = 1$, deg $t_1 = 4$ and deg $i_1 = -3$. For details of the construction of the small model and its equivalence to the Cartan model, see [A, M].

Because, typically, some or all i_j in the small model are inner products with multivectors, $d_G = d - \partial$ is not a derivation in the non–abelian case: so the product in the small model is not the obvious one. Again, see [A, M] for more details. This, however, is not a problem here since the product in the minimal Hirsch–Brown model is not the obvious one either.

There are two operators which play an important role in our description of the minimal Hirsch–Brown model and its relation to the small model (which coincides with the Cartan model when G is a torus).

Definition (2.1). Let $P = (I \otimes d^*G)\partial$ and $Q = \partial(I \otimes d^*G)$, where G is Green's operator. More briefly, $P = d^*G\partial$ and $Q = \partial d^*G$.

Since d^* and G commute, $PQ = QP = 0$.

3. The Minimal Hirsch–Brown Model.

We begin with the following abbreviations.

Definitions (3.1). (1) On $R_G \otimes \Omega_{inv}(M)$, let $\varphi = I - P$ and $\psi = I - Q$, where $P = d^*G\partial$ and $Q = \partial d^*G$, as above.

(2) In $\Omega_{inv}(M)$, let $\mathcal{H}$ be the subspace of harmonic forms, $B = im(d)$ be the boundaries and $E = im(d^*)$ be the coboundaries. So, in the Hodge decomposition, $\Omega_{inv}(M) = \mathcal{H} \oplus B \oplus E$.

Restricting ψ to $R_G \otimes B$, gives the following commutative diagram.

Lemma (3.2). The following diagram commutes, where the top arrow is the inclusion.

$$\begin{tikzcd} R_G \otimes E \arrow[r] \arrow[d, "d"'] & R_G \otimes \Omega_{inv}(M) \arrow[d, "d_G"] \\ R_G \otimes B \arrow[r, "\psi"'] & R_G \otimes \Omega_{inv}(M) \end{tikzcd}$$

Proof. $\psi d | R_G \otimes E = (d - \partial d^* dG) | R_G \otimes E = d_G | R_G \otimes E$, because $d^* dG$ is the identity on E.

(To put it another way, $d_G d^* G | R_G \otimes B = \psi | R_G \otimes B$.) $\square$

Now we define a new differential, $\overline{D}$ on $R_G \otimes \Omega_{inv}$ and the Hirsch–Brown differential, d_{HB}, on $R_G \otimes \mathcal{H}$.

Definitions (3.3). (1) On $R_G \otimes \Omega_{inv}(M)$, set $\overline{D} = \psi^{-1} d_G \psi$.

(2) On $R_G \otimes \mathcal{H}$, put $d_{HB} = (I \otimes H)\overline{D} | R_G \otimes \mathcal{H}$.

It is clear that $\overline{D}^2 = 0$; that $d_{HB}^2 = 0$, too, follows from the next lemma.

Lemma (3.4). The following diagram commutes.

$$\begin{tikzcd} R_G \otimes \Omega_{inv}(M) \arrow[r, "\psi^{-1}"] \arrow[d, "d_G"'] & R_G \otimes \Omega_{inv}(M) \arrow[r, "I\otimes H"] \arrow[d, "\overline{D}"'] & R_G \otimes \mathcal{H} \arrow[d, "d_{HB}"] \\ R_G \otimes \Omega_{inv}(M) \arrow[r, "\psi^{-1}"'] & R_G \otimes \Omega_{inv}(M) \arrow[r, "I\otimes H"'] & R_G \otimes \mathcal{H} \end{tikzcd}$$

Proof. It is enough to show that, for any $\alpha \in B \oplus E$, $(I \otimes H)\overline{D}(\alpha) = 0$. This follows from the next lemma. $\square$

Lemma (3.5). (1) On $R_G \otimes \mathcal{H}$, $\overline{D} = -\psi^{-1}\partial = -\partial\varphi^{-1}$.

(2) On $R_G \otimes B$, $\overline{D} = 0$.

(3) On $R_G \otimes E$, $\overline{D} = d$.

Proof. (1) On $R_G \otimes \mathcal{H}$, $d = 0$ and $Q = 0$: so $\overline{D} = \psi^{-1}d_G\psi = \psi^{-1}d_G = -\psi^{-1}\partial$. And $Q\partial = \partial P$.

(2) From the proof of Lemma (3.2), $\psi | R_G \otimes B = d_G d^* G | R_G \otimes B$. Hence $d_G \psi | R_G \otimes B = 0$.

(3) On $R_G \otimes E$, $Q = 0$; and so, on $R_G \otimes E$, $\overline{D} = \psi^{-1}d_G = \psi^{-1}(\psi d)$, by Lemma (3.2) $\square$

Definition (3.6). The differential R_G–module, $(R_G \otimes \mathcal{H}, d_{HB}) = (H^*(BG;\mathbb{R}) \otimes H^*(M;\mathbb{R}), d_{HB})$ is called the minimal Hirsch–Brown model for $H^*_G(M;\mathbb{R})$. That it computes $H^*_G(M;\mathbb{R})$ follows from the next lemma.

Lemma (3.7). $H(R_G \otimes \mathcal{H}, d_{HB}) \cong H^*_G(M;\mathbb{R})$, where the H on the left means (co)homology with respect to the differential d_{HB}.

Indeed, $(I \otimes H)\psi^{-1} = (1 \otimes H)(I - Q)^{-1}$ is a homotopy equivalence of differential R_G–modules.

Proof. Since ψ is an isomorphism, ψ^{-1} induces an isomorphism on cohomology . And, by Lemma (3.5) (2) and (3), $R_G \otimes (B \oplus E) = \ker(I \otimes H)$ is acyclic with respect to $\overline{D}$. So $I \otimes H$ also induces an isomorphism in cohomology. This proves the first statement of the lemma.

Since all the differential R_G–modules involved are free, so is the mapping cone of $(I \otimes$

$H)\psi^{-1}$. Thus the second statement of the lemma follows from [A, P], Remark (B.1.10), Proposition (B.1.11) and Proposition (B.1.7) $\square$

The Hirsch–Brown differential can be written in a very useful way as we show next.

Theorem (3.8). *On $R_G \otimes \mathcal{H}$,*

$$d_{HB} = (I - P)d_G(I - P)^{-1} = \varphi d_G \varphi^{-1}.$$

Proof. Let $a \in R_G \otimes \mathcal{H}$. By Lemma (3.5)(1),

$$\begin{aligned} d_{HB}(a) &= (I \otimes H)\overline{D}(a) = -(I \otimes H)\partial\varphi^{-1}(a) \\ &= -\partial\varphi^{-1}(a) + (I \otimes \Delta G)\partial\varphi^{-1}(a), \end{aligned}$$

by the Hodge Decomposition Theorem.

In general, however,

$$(I \otimes \Delta G)\partial = d^*Gd\partial + dd^*G\partial = dP - Pd,$$

where, as usual, we have abbreviated $(I \otimes d^*dG)\partial, (I \otimes d)P$, et cetera, by $d^*dG\partial$, dP, et cetera. I.e., $[d, P] = \Delta G\partial$.

$$\begin{aligned} \text{So } \Delta G\partial\varphi^{-1}(a) &= dP\varphi^{-1}(a) - Pd\varphi^{-1}(a) \\ &= d(I - \varphi)\varphi^{-1}(a) - Pd\varphi^{-1}(a) \\ &= d\varphi^{-1}(a) - Pd\varphi^{-1}(a), \text{ since } d(a) = 0. \end{aligned}$$

Finally, then, $d_{HB}(a) = -\partial\varphi^{-1}(a) + \varphi d\varphi^{-1}(a) = \varphi d_G\varphi^{-1}(a)$, since $P\partial = 0$. $\square$

Corollary (3.9). The following diagram commutes, where i_H is the inclusion.

$$\begin{array}{ccccc} R_G \otimes \mathcal{H} & \xrightarrow{i_H} & R_G \otimes \Omega_{inv}(M) & \xrightarrow{\varphi^{-1}} & R_G \otimes \Omega_{inv}(M) \\ {\scriptstyle d_{HB}}\downarrow & & \downarrow{\scriptstyle \varphi d_G \varphi^{-1}} & & \downarrow{\scriptstyle d_G} \\ R_G \otimes \mathcal{H} & \xrightarrow[i_H]{} & R_G \otimes \Omega_{inv}(M) & \xrightarrow[\varphi^{-1}]{} & R_G \otimes \Omega_{inv}(M) \end{array}$$

Furthermore, $(I \otimes H)\psi^{-1}\varphi^{-1}i_H = I$.

Proof. Since $PQ = QP = 0$, $\psi^{-1}\varphi^{-1} = I + P + Q + (P+Q)^2 + \cdots$. And $HP = 0$ and $QH = 0$. So $(I \otimes H)\psi^{-1}\varphi^{-1}i_H = I$. $\square$

Remark (3.10). Corollary (3.9) shows that $(I \otimes H)\psi^{-1}$ is a fibration and $\varphi^{-1}i_H$ is a cofibration by [A, P], Proposition (B.1.5.) and Proposition (B.1.4.).

Theorem (3.8) also reproves the main result of [A2]. Let $i : M \to M_G$ be the inclusion of a fibre in the Borel construction bundle $M_G \longrightarrow BG$.

Corollary (3.11). Suppose that $i^* : H^*_G(M;\mathbb{R}) \to H^*(M;\mathbb{R})$ is surjective. Let $\alpha \in \Omega_{inv}(M)$ be a harmonic form. Then α has a canonical equivariant extension; namely

$$d_G(I-P)^{-1}(\alpha) = 0\,.$$

Proof. Since i^* is surjective, it follows that

$$H^*_G(M;\mathbb{R}) \cong R_G \otimes H^*(M;\mathbb{R})$$

as a R_G–module. Hence $d_{HB} = 0$. Thus, by Theorem (3.8), $d_G(I-P)^{-1}(\alpha) = 0$, for all $\alpha \in \mathcal{H}$. $\square$

The operators used above, P, Q, H, Δ, G, for example, are not multiplicative, and nor is the small model itself when the Lie group is non–abelian. So, in general, even for torus actions, it is not easy to describe the product structure in the minimal Hirsch–Brown model; and, even when d_{HB} is zero, the product on $R_G \otimes \mathcal{H}$ is usually twisted. Nevertheless, for torus actions when d_{HB} is zero, we can describe the product in $H^*_G(M;\mathbb{R})$.

Definition (3.12). (1) As in [A2], we shall use CEF to mean that there is a cohomology extension of the fibre, that is,

$$i^* : H^*_G(M;\mathbb{R}) \longrightarrow H^*(M;\mathbb{R})$$

is surjective. When G is a torus, this implies that

$$j^* : H^*_G(M;\mathbb{R}) \longrightarrow H^*_G(M^G;\mathbb{R})$$

is injective where $j : M^G \to M$ is the inclusion of the fixed point set. (See, e.g., [A, P], Section (3.1).) And, for any compact connected G, as noted above, CEF implies that

$$H^*_G(M;\mathbb{R}) \cong H^*(BG;\mathbb{R}) \otimes H^*(M;\mathbb{R})$$

as R_G–modules. Thus CEF implies that $d_{HB} = 0$.

Note that, when using cohomology with coefficients in an abelian group that is not a field, then one says that there is a CEF if i^* has a right inverse. See, e.g., [S], Chap.5, Sec.7. This is, of course, equivalent to the surjectivity of i^* when using field coefficients.

(2) Let $\widetilde{\wedge}$ denote the product in the minimal Hirsch–Brown model. In particular, for $\alpha, \beta \in \mathcal{H} \subseteq \Omega_{inv}(M)$, $\alpha \widetilde{\wedge} \beta$ is the product of α and β in $R_G \otimes \mathcal{H}$, whereas, of course, $\alpha \wedge \beta$ is the product in $\Omega_{inv}(M)$.

(3) For $\alpha \in \Omega_{inv}(M)$, abbreviate $(I-P)^{-1}(\alpha) = \varphi^{-1}(\alpha)$ by $\widehat{\alpha}$.

We now have the following description of the cup–product in $H^*_G(M;\mathbb{R})$ in the CEF case when G is a torus. Of course,

$$H^*_G(M;\mathbb{R}) \cong R_G \otimes \mathcal{H};$$

and so it is enough to describe $\widetilde{\wedge}$. Indeed, since $H^*_G(M;\mathbb{R})$ is a R_G–algebra, it is enough to describe $\widetilde{\wedge}$ on $\mathcal{H}$.

Proposition (3.13). Suppose that G is a torus and that there is a CEF. Then, for $\alpha, \beta \in \mathcal{H}$,

$$\alpha \widetilde{\wedge} \beta = (I \otimes H)(1-Q)^{-1}\big(\widehat{\alpha}\widehat{\beta}\big).$$

Proof. Let $\theta = (I \otimes H)(1-Q)^{-1}$. Since $d_{HB} = 0$, $\widehat{\alpha}$ and $\widehat{\beta}$ are cycles in $(R_G \otimes \Omega_{inv}(M), d_G)$ by Corollary (3.11). And, since G is a torus, $(R_G \otimes \Omega_{inv}(M), d_G)$ is the Cartan model and, thus, multiplicative. Hence the product $\widehat{\alpha}\widehat{\beta}$ in $R_G \otimes \Omega_{inv}(M)$ represents $[\widehat{\alpha}]\big[\widehat{\beta}\,\big]$, the product in $H^*_G(M;\mathbb{R})$. And, although θ is not multiplicative, θ^* is an isomorphism; and so we have the following.

$$\begin{aligned} \alpha \widetilde{\wedge} \beta &= \theta^*([\widehat{\alpha}])\theta^*\big(\big[\widehat{\beta}\,\big]\big), \text{since } d_{HB} = 0 \\ &= \theta^*\big([\widehat{\alpha}]\big[\widehat{\beta}\,\big]\big) = \theta^*\big(\big[\widehat{\alpha}\widehat{\beta}\,\big]\big) \\ &= \big[\theta\big(\widehat{\alpha}\widehat{\beta}\big)\big] = \theta\big(\widehat{\alpha}\widehat{\beta}\big), \text{again because } d_{HB} = 0 \quad \square \end{aligned}$$

Remarks (3.14). (1) Since $\widehat{\alpha} = (I-P)^{-1}(\alpha)$, for $\alpha, \beta \in \mathcal{H}$, under the conditions of Proposition (3.13),

$$\alpha \widetilde{\wedge} \beta = H(\alpha \wedge \beta) \text{ modulo } \overline{R}_G \otimes \mathcal{H},$$

where $\overline{R}_G$ is the augmentation ideal of elements of positive degree in R_G. And, of course, $H(\alpha \wedge \beta)$ is the product of α and β in $\mathcal{H} \cong H^*(M;\mathbb{R})$.

(2) If M is a closed symplectic manifold and the action of G (any compact connected Lie group) is symplectic, then there is a CEF if and only if the action is Hamiltonian. This follows largely from results of Frankel ([F]).

(3) An argument very similar to the proof of Theorem (3.8) shows that for any $\alpha \in \Omega_{inv}(M)$,

$$(I-P)d_G(I-P)^{-1}(\alpha) = (I \otimes H)d_G(I-P)^{-1}(\alpha) + d\alpha.$$

(Briefly, $Hd_G(\widehat{\alpha}) = -H\partial(\widehat{\alpha}) = -\partial(\widehat{\alpha}) + \Delta G\partial(\widehat{\alpha}) = -\partial(\widehat{\alpha}) + dP(\widehat{\alpha}) - Pd(\widehat{\alpha}) = -\partial(\widehat{\alpha}) + d(\widehat{\alpha} - \alpha) - Pd(\widehat{\alpha}) = d_G\widehat{\alpha} - Pd_G(\widehat{\alpha}) - d\alpha$.)

(4) Under the conditions of Proposition (3.13), from Corollary (3.9), it follows similarly that, for any $\alpha, \beta \in \mathcal{H}$, there is $\gamma \in R_G \otimes \Omega_{inv}(M)$ such that

$$\varphi^{-1} i_H(\alpha \widetilde{\wedge} \beta) = \widehat{\alpha}\widehat{\beta} + d_G\gamma.$$

(5) Similar results hold for products of three or more elements.

4. An Example. Let M be a closed symplectic $2n$–manifold with symplectic form ω. Suppose that a compact connected Lie group, G, is acting on M in a Hamiltonian way. Then we may choose an invariant Riemannian metric on M that is compatible with ω. See, e.g., [M, S], Lemma 5.49. So, if r is the metric, and V_1 and V_2 are any two vector fields on M, then $r(V_1, V_2) = \omega(V_1, JV_2)$, where J is an invariant compatible almost–complex structure on M. It follows that ω^j is harmonic for $0 \leq j \leq n$; and

$$*\left(\frac{\omega^j}{j!}\right) = \frac{\omega^{n-j}}{(n-j)!},$$

for $0 \le j \le n$. In particular, $\frac{\omega^n}{n!}$ is the volume form.

As remarked above, it follows from the results of Frankel ([F]) that, in the Hamiltonian case, M has a CEF; and so, in the minimal Hirsch–Brown model, $d_{HB} = 0$. Thus the remaining problem is to determine the product structure in $H_G^*(M;\mathbb{R})$. In this section we shall do this in the familiar situation where $G = S^1$ and $M = \mathbb{C}P^n$. The results are not new, although they may be assembled in a somewhat novel way.

First, however, consider a Hamiltonian action of $G = S^1$ on any closed symplectic manifold (M, ω). Let μ be the moment map; and suppose that μ has been chosen to have average value zero on M: i.e., $\int_M \mu \frac{\omega^n}{n!} = 0$. Let V be the vector field defined by the circle action : so, for any $x \in M$,

$$V_x = \frac{d}{du} \exp(2\pi i u)x|_{u=0}.$$

In the Cartan model, then, the differential $d_G = d - t i_V$, where $t \in H^2(BG;\mathbb{R})$ is the polynomial generator. In the Hodge decomposition $\mu = d^* dG\mu$, since the harmonic part, $H(\mu)$, is the average value. Thus $P(\omega) = t\mu$, because $d^* G i_V(\omega) = d^* G d\mu = \mu$. Hence $\widehat{\omega} = \omega + t\mu$, the standard equivariant extension of ω.

Now let $M = \mathbb{C}P^n$ with symplectic form ω and Hamiltonian action of $G = S^1$. Let μ be the moment map; but we shall not assume that $H(\mu) = 0$. Let $\overline{w} = [\omega + t\mu]_G \in H_G^2(M;\mathbb{R})$. The product structure in $H_G^*(M;\mathbb{R})$ is completely determined by expressing $\overline{w}^{n+1}$ in terms of lower powers of $\overline{w}$. Let $\overline{w}^{n+1} = \sum_{i=1}^{n+1} c_i \overline{w}^{n+1-i} t^i$, where each $c_i \in \mathbb{R}$. One way to find the c_i is the following.

For $j \ge 0$, $\overline{w}^{n+1+j} = \sum_{i=1}^{n+1} c_i \overline{w}^{n+1+j-i} t^i$. So integrating over the fibre, M, in the Borel

construction bundle $M_G \to BG$, gives

$$\binom{n+1+j}{1+j} t^{1+j} \int_M \mu^{1+j}\omega^n = \sum_{i=1}^{1+j} c_i \binom{n+1+j-i}{1+j-i} t^{1+j} \int_M \mu^{1+j-i}\omega^n .$$

$$\text{So } \binom{n+1+j}{1+j} H(\mu^{1+j}) = \sum_{i=1}^{1+j} c_i \binom{n+1+j-i}{1+j-i} H(\mu^{1+j-i}).$$

Since this holds for all $j \geq 0$, one can easily solve for each c_i in terms of the average values of the powers of μ. For example, putting $j = 0$, $c_1 = (n+1)H(\mu)$; and, putting $j = 1$,

$$c_2 = \binom{n+2}{2} H(\mu^2) - (n+1)^2 H(\mu)^2.$$

Equally, one can solve for each $H(\mu^j)$ in terms of $c_1, \dots, c_j$. This is reasonable because there are other familiar ways to find the $c_i s$. Let the fixed point set $M^G = \bigcup\limits_{i=1}^{s} F_i$, where the component F_i has dimension $2r_i$. By the equality of Euler characteristics, $\sum\limits_{i=1}^{s}(r_i+1) = n+1$. Let ν_i be the value of μ on F_i; and let $\mu_j = \nu_i$ for $\sum\limits_{k=1}^{i-1}(r_k+1)+1 \leq j \leq \sum\limits_{k=1}^{i}(r_k+1)$. So the distinct values of μ appear with multiplicity, each ν_i appearing with multiplicity $r_i + 1$. (If M^G is finite, then $s = n+1$, and $\mu_i = \nu_i$ for $1 \leq i \leq n+1$.) In terms of these values we have

$$\prod_{i=1}^{s} (\overline{w} - \nu_i t)^{r_i+1} = \prod_{i=1}^{n+1} (\overline{w} - \mu_i t) = 0.$$

This follows from the Localization Theorem of Borel, Hsiang and Quillen. For details of this example see [H], Theorem (IV.3) or, e.g., [A1], Example (3.12). Thus, for $1 \leq i \leq n+1$, $c_i = (-1)^{i+1}\sigma_i$, where σ_i is the ith elementary symmetric polynomial in $\mu_1, \dots, \mu_{n+1}$.

Now suppose that M^G is finite: so $s = n+1$ and each $r_i = 0$. Let $U_i = \prod\limits_{j \neq i}(\overline{w} - \mu_j t)$. So U_i restricts to $\prod\limits_{j\neq i}(\mu_i - \mu_j)t^n$ at F_i and zero at all the other fixed points. Let the equivariant

Euler class at F_i be $\varepsilon_i t^n$, normalized so that ε_i is an integer (the product of the weights). Integrating U_i over the fibre gives

$$\int_M \omega^n = \frac{1}{\varepsilon_i} \prod_{j \neq i} (\mu_i - \mu j) \tag{4.1}$$

by the Integration Formula ([A, B], (3.8)). (See [B], Chapter VIII, Theorem 5.5 (based on the original example of W.–Y. Hsiang), [P] for many related results, or [A1], Example (4.16) for an elementary treatment.)

Meanwhile the Duistermaat–Heckman formula gives

$$\int_M e^{\mu t} \frac{\omega^n}{n!} = \sum_{i=1}^{n+1} \frac{e^{\mu_i t}}{\varepsilon_i t^n}.$$

Thus

$$\binom{n+j}{j} H(\mu^j) = \sum_{i=1}^{n+1} \frac{\mu_i^{n+j}}{\prod_{j \neq i} (\mu_i - \mu_j)}.$$

The last formula is homogeneous in μ and, hence, not sensitive to such matters as how on parametrizes the circle ($\exp(2\pi i t)$ versus $\exp(it)$), what sign convention one uses for $\mu (d\mu = i_V(\omega)$ versus $d\mu = -i_V(\omega))$ or the sign of $t (d_G = d - t i_V$ versus $d_G = d + t i_V)$.

Given a particular linear action, one can use (4.1) to find the $\mu_i s$, and, hence, the $c_i s$. For example, let S^1 act on $\mathbb{C}P^2$ by $z[z_0, z_1, z_2] = [z_0, z^a z_1, z^b z_2]$, where a and b are integers such that $0 < a < b$. Let $\int_M \omega^2 = A$. Choose μ so that $H(\mu) = 0$. Then one gets

$$\begin{aligned} 6H(\mu^2) &= \frac{A}{3}(a^2 - ab + b^2) = c_2 \\ \text{and } 10H(\mu^3) &= \frac{A\sqrt{A}}{27}\,(2a^3 - 3a^2 b - 3ab^2 + 2b^3) = c_3\,. \end{aligned}$$

Thus

$$\overline{w}^3 = \frac{1}{3}(a^2 - ab + b^2) A \overline{w} t^2 + \frac{1}{27}(2a^3 - 3a^2 b - 3ab^2 + 2b^3) A\sqrt{A}\, t^3\,.$$

($c_3 = \mu_1 \mu_2 \mu_3 = \frac{-A\sqrt{A}}{27}(a+b)(2a-b)(2b-a)$, for example.)

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